ON A PARTITION PROBLEM OF CANFIELD AND WILF

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To the memory of John Selfridge

Abstract. Let A and M be nonempty sets of positive integers. A partition of the positive integer n with parts in A and multiplicities in M is a representation of n in the form $n = \sum_{a\in A} m_a a$ where $m_a \in M \cup \{0\}$ for all $a \in A$, and $m_a \in M$ for only finitely many a. Denote by $p_{A,M}(n)$ the number of partitions of n with parts in A and multiplicities in M. It is proved that there exist infinite sets A and M of positive integers whose partition function $p_{A,M}$ has weakly superpolynomial but not superpolynomial growth. The counting function of the set A is $A(x) = \sum_{a\in A, a\leq x} 1$. It is also proved that $p_{A,M}$ must have at least weakly superpolynomial growth if M is infinite and $A(x) \gg \log x$.

1. Partition problems with restricted multiplicities

Let A be a nonempty set of positive integers. A *partition of n with parts in A* is a representation of n in the form

$$n = \sum_{a\in A} m_a a$$

where $m_a \in \mathbf{N} \cup \{0\}$ for all $a \in A$, and $m_a \in \mathbf{N}$ for only finitely many a. Let $p_A(n)$ denote the number of partitions of n with parts in A. If $\gcd(A) = d > 1$, then $p_A(n) = 0$ for all n not divisible by d, and so $p_A(n) = 0$ for infinitely many positive integers n. If $p_A(n) \geq 1$ for all sufficiently large n, then $\gcd(A) = 1$.

If $A = \{a_1, \ldots, a_k\}$ is a set of k relatively prime positive integers, then Schur [7] proved that

$$p_A(n) \sim \frac{n^{k-1}}{(k-1)!a_1a_2\cdots a_k}. \tag{1}$$

Nathanson [5] gave a simpler proof of the more precise result:

$$p_A(n) = \frac{n^{k-1}}{(k-1)!a_1a_2\cdots a_k} + O\left(n^{k-2}\right). \tag{2}$$

An arithmetic function is a real-valued function whose domain is the set $\mathbf{N}$ of positive integers. An arithmetic function f has *polynomial growth* if there is a positive integer k and an integer $N_0(k)$ such that $f(n) \leq n^k$ for all $n \geq N_0(k)$. Equivalently, f has polynomial growth if

$$\limsup_{n\to\infty} \frac{\log f(n)}{\log n} < \infty.$$

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An arithmetic function f has *superpolynomial growth* if

$$\lim_{n\to\infty} \frac{\log f(n)}{\log n} = \infty.$$

The asymptotic formula (1) implies the following result of Nathanson [4, Theorem 15.2, pp. 458–461].

Theorem 1. *If A is an infinite set of integers and* $\gcd(A) = 1$, *then $p_A(n)$ has superpolynomial growth.*

Canfield and Wilf [2] studied the following variation of the classical partition problem. Let A and M be nonempty sets of positive integers. A *partition of n with parts in A and multiplicities in M* is a representation of n in the form

$$n = \sum_{a\in A} m_a a$$

where $m_a \in M \cup \{0\}$ for all $a \in A$, and $m_a \in M$ for only finitely many a. We denote by $p_{A,M}(n)$ the number of partitions of n with parts in A and multiplicities in M. Note that $p_{A,M}(0) = 1$ and $p_{A,M}(n) = 0$ for all $n < 0$.

Let A and M be infinite sets of positive integers such that $p_{A,M}(n) \geq 1$ for all sufficiently large n. Canfield and Wilf ("Unsolved problem 1" in [2]) asked, "Must $p_{A,M}(N)$ then be of superpolynomial growth?" We prove that the answer is "no."

2. Weakly superpolynomial functions

Polynomial and superpolynomial growth functions were first studied in connection with the growth of finitely and infinitely generated groups (cf. Grigorchuk and Pak [3], Nathanson [6]). Growth functions of groups are always strictly increasing, but even strictly increasing functions that do not have polynomial growth are not necessarily superpolynomial.

We shall call an arithmetic function *weakly superpolynomial* if it does not have polynomial growth. Equivalently, the function f is weakly superpolynomial if for every positive integer k there are infinitely many positive integers n such that $f(n) > n^k$. The partition functions that will be constructed in this paper are weakly superpolynomial but not superpolynomial.

We note that an arithmetic function f is weakly superpolynomial but not superpolynomial if and only if

$$\limsup_{n\to\infty} \frac{\log f(n)}{\log n} = \infty$$

and

$$\liminf_{n\to\infty} \frac{\log f(n)}{\log n} < \infty.$$

In this section we construct a strictly increasing arithmetic function that is weakly superpolynomial but not polynomial.

Let $(n_k)_{k=1}^{\infty}$ be a sequence of positive integers such that $n_1 = 1$ and

$$n_{k+1} > 2n_k^k$$

for all $k \geq 1$. We define the arithmetic function

$$f(n) = n_k^k + (n - n_k) \qquad \text{for } n_k \leq n < n_{k+1}.$$

This function is strictly increasing because

$$n_k^k - n_k \leq n_{k+1}^{k+1} - n_{k+1}$$

for all $k \geq 1$. We have

$$\lim_{k\to\infty} \frac{\log f(n_k)}{\log n_k} = \lim_{k\to\infty} \frac{k \log n_k}{\log n_k} = \infty$$

and so

$$\limsup_{n\to\infty} \frac{\log f(n)}{\log n} = \infty.$$

Therefore, the function f does not have polynomial growth.

For every positive integer n there is a positive integer k such that $n_k \leq n < n_{k+1}$. Then $f(n) = n + n_k^k - n_k \geq n$ and so

$$\liminf_{n\to\infty} \frac{\log f(n)}{\log n} \geq 1. \tag{3}$$

The inequalities

$$f(n_{k+1} - 1) = n_k^k + (n_{k+1} - 1 - n_k) < \frac{3n_{k+1}}{2}$$

and

$$n_{k+1} - 1 > \frac{n_{k+1}}{2}$$

imply that

$$1 < \frac{\log f(n_{k+1} - 1)}{\log(n_{k+1} - 1)} < \frac{\log(3n_{k+1}/2)}{\log(n_{k+1}/2)} = 1 + \frac{\log 3}{\log(n_{k+1}/2)}$$

and so

$$\lim_{k\to\infty} \frac{\log f(n_{k+1} - 1)}{\log(n_{k+1} - 1)} = 1.$$

Therefore,

$$\liminf_{n\to\infty} \frac{\log f(n)}{\log n} \leq 1. \tag{4}$$

Combining (3) and (4), we obtain

$$\liminf_{n\to\infty} \frac{\log f(n)}{\log n} = 1.$$

Thus, the function f has weakly superpolynomial but not superpolynomial growth.

3. Weakly superpolynomial partition functions

Theorem 2. *Let a be an integer, $a \geq 2$, and let $A = \{a^i\}_{i=0}^{\infty}$. Let M be an infinite set of positive integers such that M contains $\{1, 2, \ldots, a-1\}$ and no element of M is divisible by a. Then $p_{A,M}(n) \geq 1$ for all $n \in \mathbf{N}$, and $p_{A,M}(n) = 1$ for all $n \in A$. In particular, the partition function $p_{A,M}$ does not have superpolynomial growth.*

Proof. Every positive integer n has a unique a-adic representation, and so $p_{A,M}(n) \geq 1$ for all $n \in \mathbf{N}$.

We shall prove that the only partition of a^r with parts in A and multiplicities in M is $a^r = 1 \cdot a^r$. If there were another representation, then it could be written in the form

$$a^r = \sum_{i=1}^{k} m_i a^{j_i}$$

where $k \geq 2$, $m_i \in M$ for $i = 1, \ldots, k$, and $0 \leq j_1 < j_2 < \cdots < j_k < r$. Then

$$a^{r-j_1} = m_1 + a \sum_{i=2}^{k} m_i a^{j_i - j_1 - 1}.$$

We have $j_i - j_1 - 1 \geq 0$ for $i = 2, \ldots, k$, and so m_1 is divisible by a, which is absurd. Therefore, $p_{A,M}(a^r) = 1$ for all $r \geq 0$. It follows that

$$\liminf_{n \to \infty} \frac{\log p_{A,M}(n)}{\log n} = \liminf_{r \to \infty} \frac{\log p_{A,M}(a^r)}{\log a^r} = 0$$

and so the partition function $p_{A,M}$ is not superpolynomial. □

Theorem 3. *Let A and M be infinite sets of positive integers, and let $p_{A,M}(n)$ denote the number of partitions of n with parts in A and multiplicities in M. If $A(x) \geq c \log x$ for some $c > 0$ and all $x \geq x_0(A)$, then for every positive integer k there exist infinitely many integers n such that*

$$p_{A,M}(n) \geq n^k.$$

In particular, the partition function $p_{A,M}$ is weakly superpolynomial.

Proof. Let $x \geq 1$ and let

$$A(x) = \sum_{\substack{a \in A \\ a \leq x}} 1 \qquad \text{and} \qquad M(x) = \sum_{\substack{m \in M \\ m \leq x}} 1$$

denote the counting functions of the sets A and M, respectively. If $n \leq x$ and $n = \sum_{a \in A} m_a a$ is a partition of n with parts in A and multiplicities in M, then $a \leq x$ and $m_a \leq x$, and so

$$\max \{p_{A,M}(n) : n \leq x\} \leq \sum_{n \leq x} p_{A,M}(n) \leq (M(x)+1)^{A(x)}. \tag{5}$$

Conversely, if the integer n can be represented in the form $n = \sum_{a \in A} m_a a$ with $a \leq x$ and $m_a \leq x$, then $n \leq x^2 A(x)$ and so

$$\sum_{n \leq x^2 A(x)} p_{A,M}(n) \geq (M(x)+1)^{A(x)} > M(x)^{A(x)}.$$

Choose an integer n_x such that

$$p_{A,M}(n_x) = \max \left\{ p_{A,M}(n) : n \leq x^2 A(x) \right\}.$$

Then $n_x \leq x^3$ and

$$M(x)^{A(x)} < \sum_{n \leq x^2 A(x)} p_{A,M}(n) \leq \left(x^2 A(x) + 1\right) p_{A,M}(n_x) \leq 2x^3 p_{A,M}(n_x)$$

and so, for all $x \geq x_0(A)$,

$$p_{A,M}(n_x) > \frac{M(x)^{A(x)}}{2x^3} \geq \frac{M(x)^{c \log x}}{2x^3}.$$

Let k be a positive integer. Because the set M is infinite, there exists $x_1(A,k) \geq x_0(A)$ such that, for all $x \geq x_1(A,k)$, we have

$$\log M(x) > \frac{\log 2}{c \log x} + \frac{3k+3}{c}$$

and so

$$p_{A,M}(n_x) > x^{3k} \geq n_x^k.$$

We shall iterate this process to construct inductively an infinite set of integers $\{n_{x_i} : i = 1, 2, \ldots\}$ such that

$$p_{A,M}(n_{x_i}) > n_{x_i}^k$$

for all i. Let $r \geq 1$, and suppose that the integers $n_{x_1}, \ldots, n_{x_r}$ have been constructed. Choose x_{r+1} so that

$$x_{r+1}^{3k} > \left(M\left(x_i^2 A(x_i)\right) + 1\right)^{A\left(x_i^2 A(x_i)\right)}$$

for all $i = 1, \ldots, r$, and let $n_{x_{r+1}}$ be the integer constructed according to procedure above. Replacing x with $x_i^2 A(x_i)$ in inequality (5), we see that

$$p\left(n_{x_i}\right) \leq \left(M\left(x_i^2 A(x_i)\right) + 1\right)^{A\left(x_i^2 A(x_i)\right)}$$

and so

$$p\left(n_{x_{r+1}}\right) > x_{r+1}^{3k} > p\left(n_{x_i}\right)$$

for $i = 1, \ldots, r$, and so $n_{x_{r+1}} \neq n_{x_i}$ for $i = 1, \ldots, r$. This completes the induction and the proof. □

Theorem 4. *The partition function for the sets A and M constructed in Theorem 2 is weakly superpolynomial.*

Proof. For $a \geq 2$, the counting function for the set $A = \{a^i\}_{i=1}^{\infty}$ is $A(x) = [\log x] + 1 > \log x$, and the result follows from Theorem 3. □

4. Open problems

(1) Do there exist infinite sets A and M of positive integers such that $p_{A,M}(n) \geq 1$ for all sufficiently large n and $p_{A,M}(n)$ has polynomial growth? This is, perhaps, the intended statement of Canfield and Wilf's "Unsolved problem 1."

(2) By Theorem 3, if the partition function $p_{A,M}$ has polynomial growth, then the set A must have sub-logarithmic growth, that is, $A(x) \gg \log x$ is impossible.

   (a) Let $A = \{k!\}_{k=1}^{\infty}$. Does there exist an infinite set M of positive integers such that $p_{A,M}(n) \geq 1$ for all sufficiently large n and $p_{A,M}$ has polynomial growth?

   (b) Let $A = \left\{k^k\right\}_{k=1}^{\infty}$. Does there exist an infinite set M of positive integers such that $p_{A,M}(n) \geq 1$ for all sufficiently large n and $p_{A,M}$ has polynomial growth?

(3) Let A be an infinite set of positive integers and let $M = \mathbf{N}$. Bateman and Erdős [1] proved that the partition function $p_A = p_{A,\mathbf{N}}$ is eventually strictly increasing if and only if $\gcd(A \setminus \{a\}) = 1$ for all $a \in A$. It would be interesting to extend this result to partition functions with restricted multiplicities: Determine a necessary and sufficient condition for infinite

sets A and M of positive integers to have the property that $p_{A,M}(n) < p_{A,M}(n+1)$ or $p_{A,M}(n) \leq p_{A,M}(n+1)$ for all sufficiently large n.

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